CS522 - American Options

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0.1 Complementarity Formulation for American Options

The difference between European and American options is that American options can be exercised early. As stated before, this, together with the absence of arbitrage opportunities, imposes certain relations between the prices of American and European options.

In the following, we will examine the case of American puts on stocks that do not pay dividends. The results we obtain and the methodology we develop can be extended to options on stocks paying a constant dividend yield, or paying discrete dividends. Recall that the case of American calls on stocks paying no dividends is trivial; the option will never be exercised early; hence, its value must be equal to that of the European call with same maturity and strike price.

In the following, we will use $p(t, K)$, $P(t, K)$ to denote the value of a European and American put option with strike price K and maturity t, respectively. We have already established the following:[1]

$$
\begin{aligned}
&P(t, S) \geqslant p(t, S) \\
&P(t, S) \geqslant \max(K - S, 0) \\
&P(t, 0) = K \\
&p(t, 0) = Ke^{-r(T-t)} \\
&\lim_{S \to \infty} p(t, S) = 0
\end{aligned}
$$

We will accept without further ado that $p(t, S)$ is continuous with respect to S.

Let us analyze the two extremes of S.

If S ever becomes 0, it will never change. In this case, both the European and American puts will be exercised for sure. The European option can only be exercised at maturity, thus $p(t, 0)$ will be the time-t discounted value of the K dollars that will be received at time T. An American put holder has no reason not to exercise immediately, as any delay would mean that interest that could have been earned on the strike price K will be lost, while the payoff will never increase above K. Hence, the time-0 value of the American put will be K.

If S becomes very large, the value of European puts will decrease toward 0. This is because large stock prices make it unlikely that prices will fall under K. The price $p(0, S)$ will never be 0, however. If we assume that the volatility of the stock is non-zero, then there is always a non-zero probability that the stock price will be less than K at expiration. Since $P(t, S) \geqslant p(t, S)$, we conclude that $P(t, S) > 0$.

[1] We did this directly, or we asked the reader to establish the corresponding formulas by analogy to those for puts.

Let us now consider the instant payoff of an exercised American option $\max(K-S,0)$. If $S=0$, then $p(t,0)=K=\max(K-S,0)$, i.e. the price of the put lies on the graph of the instant payoff (and the option is exercised early). For very large S, however, we have that $p(t,S)>\max(K-S,0)=0$, i.e. the price of the put lies above the graph of the instant payoff (and the option is not exercised early). These insights allow us to conclude that there must be a point (stock price) $S_f(t)$ where the value of the American put "breaks free" from the graph of the instant payoff function $\max(K-S,0)$.[2]

Since the value of the American put never becomes 0, the graph of $p(t,S)$ can never intersect the graph of $\max(K-S,0)$ in the region $S \geqslant K$. Thus we must have that $S_f(t) \in (0,K)$.

For values of $0 \leqslant S \leqslant S_f(t)$ the American put will be exercised early; for $S > S_f(t)$, the respective put will not be exercised. We call the value $S_f(t)$ the frontier point of the early exercise region. The graph $\{(t, S_t(t)) \mid t \leqslant T\}$ represents the early exercise frontier of the American put.

0.1.1 The Black-Scholes Equation for American Options

Given a European payoff with value $V(t,S)$, we formed the portfolio

$$\mathbf{P} = V - \Delta S,$$

where $\Delta=\frac{\partial V}{\partial S}$ over an infinitesimal period of time $(t, t+dt)$. In effect, we delta-hedged our payoff. This choice eliminates all randomness in the change of V over the respective interval:

$$d\mathbf{P} = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt.$$

A simple arbitrage argument convinced us that $d\mathbf{P}$ must be equal to the money that could have been earned if we invested the value of the portfolio in the money market account $d\mathbf{P} = r\mathbf{P}dt$.

But what happens in the case of an American payoff? Clearly, the difference must lie in the possibility of early exercise. This, in turn, will have an impact on our arbitrage argument.

Let us assume that $d\mathbf{P} < r\mathbf{P}dt$. This relationship states that the return on the portfolio is less than what could be earned on the money market account. Such a relationship can not hold in the case of a European option, as nobody would be willing to pay $\mathbf{P}$ for the portfolio just to earn less than the same amount $\mathbf{P}$ would have earned on the money market account. This will depress the price of the portfolio containing the European instrument so that the equality is reestablished. If early exercise is possible, however,

[2] Remember that the value of the American put can never decrease under $\max(S-K,0)$. This is because the American option can be exercised early. As opposed to this, the value of a European option can be less that the value of the instant payoff (can you think of a simple example?).

It could be possible, of course, that the function $p(t,S)$ "breaks away" and returns several times from the graph of the instant payoff, and then "hits" it - but never crosses it! - again. We ignore this possibility here.

then the inequality can hold without implying arbitrage. If the holder would pass on the option, he could not sell it for $\mathbf{P}$, but he can just cash in the option by exercising it. After exercising the option, the holder can then invest in the money market account.

If $d\mathbf{P} = r\mathbf{P}dt$ we have the same situation as for European options. The holder of the option will earn the same over the infinitesimal interval whether he invests the money in the portfolio, or in the money market account. Hence he will not exercise early, since he might still earn more money on the option in the future.

If $d\mathbf{P} > r\mathbf{P}dt$, then one has no reason to exercise the option, as the holder can earn more on the portfolio that investing the value of the portfolio in the money market account. Any investor can borrow money from the money market account and invest it in the portfolio (i.e. buy the portfolio). At the end of the infinitesimal interval the investor liquidates the portfolio, and repays the debt to the money market account (with interest included). The investor is then left with a sure profit of $d\mathbf{P} - r\mathbf{P}dt > 0$. This is an arbitrage opportunity, which we exclude by assumption.

We conclude that in the absence of arbitrage, then, we must have that $d\mathbf{P} \leqslant r\mathbf{P}dt$.

Continuing with the derivation of the Black-Scholes equation exactly as before, we obtain that the following differential inequality must hold:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leqslant 0.$$

0.1.2 Black-Scholes Differential Inequality for American Puts

Keeping in mind that we are talking about American puts, we can state the following:

(a) In case of early exercise, i.e. if $S \leqslant S_f(t)$, we must have

$$\begin{cases} P(t,S) = K - S = \max(K-S,0) \\ \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rV < 0 \end{cases},$$

and

(b) In case early exercise is not optimal, i.e. if $S > S_f(t)$, we must have

$$\begin{cases} P(t,S) > \max(K-S,0) \\ \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0 \end{cases}.$$

In the early exercise region, the value of the put is given by the instantaneous payoff function. In the no early exercise region the solution is given by the solution to the regular Black-Scholes equation. The problem is that we can not specify a priori the position of the early exercise frontier, i.e. we can not specify function $S_f : [0,T] \to [0,K]$. This means that we must simultaneously find the value of P and determine the early exercise frontier. Such problems are called free-boundary problems.

The additional conditions that must be imposed in order to make the solution unique are as follows:

(a) Condition at expiration: $P(T,S) = \max(K-S,0)$;

(b) Boundary condition 1: $P(t,0) = K$;

(c) Boundary condition 2: $\lim_{S\to\infty} P(t,S) = 0$;

(d) Value at frontier: $P(t,S_f(t)) = K - S_f(t)$;

(e) Slope at frontier: $\frac{\partial P}{\partial S}(t,S_f(t)) = -1$.[3]

0.1.3 Coordinate and Function Changes

As before, we perform the substitutions[4]

$$\left\{\begin{array}{l} S = Ke^x \\ t = T - \frac{1}{\frac{1}{2}\sigma^2}\tau \\ u(\tau,x) = e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k+1)^2\tau}\frac{P(t,S)}{K} \end{array}\right. .$$

in the Black-Scholes differential inequality (remember that $k = \frac{r}{\frac{1}{2}\sigma^2}$).

The inequality $P(t,S) \geqslant \max(K-S,0)$ now becomes

$$u(\tau,x) \geqslant g(\tau,x) = e^{\frac{1}{4}(k+1)^2\tau}\max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0).$$

Our earlier conclusions must be restated in terms of the new variables and functions:

(a) In case of early exercise, i.e. if $x \leqslant x_f(t)$, we must have

$$\left\{\begin{array}{l} u(\tau,x) = g(\tau,x) \\ \frac{\partial u}{\partial t} > \frac{\partial^2 u}{\partial t^2} \end{array}\right. .$$

(b) In case early exercise is not optimal, i.e. if $x > x_f(t)$, we must have

$$\left\{\begin{array}{l} u(\tau,x) > g(\tau,x) \\ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} \end{array}\right. .$$

The transformed conditions that must be imposed in order to make the solution unique are as follows:

(a) Initial condition: $u(0,x) = g(0,x) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0)$;

(b) Boundary condition 1: $\lim_{x\to-\infty} u(\tau,x) = g(\tau,x)$;

[3] This condition states that the two branches of function $p(t,S)$ connect smoothly at $S = S_f$; both the function value and the first derivative is continuous. This is known as the smooth pasting condition. We will accept this condition as a fact.

[4] Note that we performed both function changes simultaneously, skipping the intermediate step.

(c) Boundary condition 2: $\lim_{x\to\infty} u(\tau, x) = 0$;

We have not provided the conditions at the early exercise frontier; as we will see shortly, these will not be needed explicitly. Can you determine the form of these conditions given the variable and function changes that we undertook?

As written, these inequalities are difficult to solve directly, primarily because we do not know where the frontier is (i.e. we do not know the value of x_f). We can, however, rewrite these conditions so that the position of the frontier becomes implicit. This formulation, known as the linear complementarity form, is as follows:

$$\begin{cases} \left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial t^2}\right)(u(\tau, x) - g(\tau, x)) = 0 \\ \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial t^2} \geqslant 0 \\ u(\tau, x) - g(\tau, x) \geqslant 0 \end{cases}.$$

Of course, we still have to impose the initial and the boundary conditions, but we do not have to know the position of the frontier anymore. Instead, we will just impose that both $u(\tau, x)$ and $\frac{\partial u}{\partial x}(\tau, x)$ are continuous.

We will accept without further proof that the linear complementarity problem is equivalent with the original formulation, i.e. it accepts a unique solution identical to that of the original problem.

Note that once we have a solution for $u(\tau, x)$, we can always find the position of the free boundary (same as the early exercise frontier). To see this, let us fix the time variable to τ_*, and let us allow x to vary between $-\infty$ and $+\infty$. As we start from very small values, the value of $u(\tau_*, x)$ will be equal to $g(\tau_*, x)$, i.e. the option will be exercised early. Before we reach the frontier, there will be points to the right of the current point x that also have this property. The frontier point corresponds to the highest value of x for which the equality still holds. In other words, $u(\tau_*, x_f) = g(\tau_*, x_f)$, but $x > x_f \Rightarrow u(\tau_*, x_f) > g(\tau_*, x_f)$. This is because the frontier point is the one where $P(t, S)$ breaks away from the value of the instant payoff function, and the same property holds for the transformed function $u(\tau, x)$.

The frontier point can be determined approximately even if the solution is only known at the discrete nodes of a grid.

0.2 Numerical Valuation of American Puts

0.2.1 Finite Differences

As before, we first divide the interval $[x_{\min}, x_{\max}]$ into an integer number of subintervals of length δx; let $N_x = \frac{x_{\max}-x_{\min}}{\delta x}$. Next, we divide the interval $[0, \tau_{\max}]$ into an integer number of subintervals of length δt; let $N_t = \frac{\tau_{\max}-0}{\delta t}$. These two steps fully discretize the domain. We introduce the notation $u_n^m = u(m\delta\tau, x_{\min} + n\delta x)$, $0 \le m \le N_t$, $0 \le n \le N_x$, to denote the values of the unknown function at the points of the resulting mesh.

0.2.2 The Crank-Nicholson Method

Recall that the error term for the Crank-Nicholson has an error term of $O((\delta t)^2)+O((\delta x)^2)$, which makes us prefer it over both the explicit and the fully implicit method. Also, this method is stable for any value of α.

We can write the following relation for all values $0 < n < N_x$ and $0 < m < N_t$:

$$-\frac{1}{2}\alpha u_{n-1}^{m+1} + (1+\alpha)u_n^{m+1} - \frac{1}{2}\alpha u_{n+1}^{m+1} = \underbrace{\frac{1}{2}\alpha u_{n-1}^{m} + (1-\alpha)u_n^{m} + \frac{1}{2}\alpha u_{n+1}^{m}}_{Z_n^m}$$

The set of equations given above can be written in matrix form as follows:

$$\underbrace{\begin{bmatrix} 1+\alpha & -\frac{1}{2}\alpha & 0 & \dots & 0 & 0 \\ -\frac{1}{2}\alpha & 1+\alpha & -\frac{1}{2}\alpha & \dots & 0 & 0 \\ 0 & -\frac{1}{2}\alpha & 1+\alpha & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1+\alpha & -\frac{1}{2}\alpha \\ 0 & 0 & 0 & \dots & -\frac{1}{2}\alpha & 1+\alpha \end{bmatrix}}_{M_{CN}} U^{m+1} = \underbrace{\begin{bmatrix} Z_1^m \\ Z_2^m \\ Z_3^m \\ \dots \\ Z_{N_x-2}^m \\ Z_{N_x-1}^m \end{bmatrix} + \frac{1}{2}\alpha \begin{bmatrix} f_0^{m+1} \\ 0 \\ 0 \\ \dots \\ 0 \\ f_{N_x}^{m+1} \end{bmatrix}}_{b_{CN}^m}$$

Here U^m denotes the column matrix $\begin{bmatrix} u_1^m & u_2^m & u_3^m & \cdots & u_{N_x-1}^m \end{bmatrix}^T$.

We used the subscript CN to denote matrices related to the Crank-Nicholson method, so that they are distinguishable from the analogous matrices we defined for the fully implicit method. Since we do not discuss the explicit and fully implicit finite difference methods here, we will drop the CN subscript. Hence, the system of equations above will be represented as $MU^{m+1} = b^m$. The absence of a superscript on matrix M emphasizes that M does not depend on time.[5]

0.2.3 Discretized Linear Complementarity Formulation

Recalling that we denoted by $g(\tau, x)$ the value of the instant payoff function in the transformed formulation $g(\tau, x) = e^{\frac{1}{4}(k+1)^2\tau}\max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0)$, we introduce the notation g_n^m for $e^{\frac{1}{4}(k+1)^2\tau_m}\max(e^{\frac{1}{2}(k+1)x_n} - e^{\frac{1}{2}(k-1)x_n}, 0)$, and G^m for the column matrix $\begin{bmatrix} g_1^m & g_2^m & g_3^m & \cdots & g_{N_x-1}^m \end{bmatrix}^T$.

With these notations, we can write the linear complementarity problem's discrete version:

$$\begin{cases} (MU^m - b^m) \bullet (U^m - G^m) = 0 \\ MU^{m+1} \geqslant b^m \\ U^{m+1} \geqslant G^{m+1} \end{cases}.$$

[5] We note in passing that this property makes it possible to factorize the matrix only once, and then reuse the factorization at every time step. While this idea is useful for determining the value of European options, for the problem of American options the issue of factorization does not arise.

The inequalities in the relations above must be understood component-wise. Note that the product denoted by $\bullet$ must be interpreted as being performed component-wise (i.e. row by row).

0.2.4 Successive Over-relaxation

For $0 < n < N_x$ and $0 < m < N_t$, relation

$$-\frac{1}{2}\alpha u_{n-1}^{m+1} + (1+\alpha)u_n^{m+1} - \frac{1}{2}\alpha u_{n+1}^{m+1} = Z_n^m,$$

can be rewritten to emphasize the value of u_n^{m+1}:

$$u_n^{m+1} = \frac{1}{1+\alpha}\left[Z_n^m + \frac{1}{2}\alpha\left(u_{n-1}^{m+1} + u_{n+1}^{m+1}\right)\right].$$

Remember that u_0^m and $u_{N_x}^m$ are known for any $m \geqslant 0$; these are the boundary conditions, and they are known.

For reasons that will become apparent shortly, we will solve this system of equations through iterative methods. Let us denote the values of u_n^m in the k^{th} iteration by $u_n^{m,k}$. The values $u_n^{m+1,0}$ will always be set to the values computed for u_n^m in the preceding step.

The following equality is obvious:

$$u_n^{m+1,k+1} = u_n^{m+1,k} + (u_n^{m+1,k+1} - u_n^{m+1,k}).$$

This equality states the obvious fact that the new approximation u_n^m is just the value of the old approximation adjusted to reflect the difference between the new approximation and the old one. If we had some independent method of estimating the $u_n^{m+1,k+1} - u_n^{m+1,k}$, we could use this equality to get updated approximations for u_n^{m+1}.

We now provide a technique that allows for such corrections; we introduce the various elements step by step.

We start with the following simple idea:

$$\begin{cases} y_n^{m+1,k+1} = \frac{1}{1+\alpha}\left[Z_n^m + \frac{1}{2}\alpha\left(u_{n-1}^{m+1,k} + u_{n+1}^{m+1,k}\right)\right] \\ u_n^{m+1,k+1} = u_n^{m+1,k} + (y_n^{m+1,k+1} - u_n^{m+1,k}) \end{cases}.$$

It can be shown that starting from $u_n^{m+1,0} = u_n^m$, $u_n^{m+1,k}$ will converge to the solution of the system of equations $MU^m = b^m$. Two simple changes, however, can make the convergence of this scheme faster.

First, we note that we compute $u_{n-1}^{m+1,k+1}$ before we compute $u_n^{m+1,k+1}$. If the values u_n^{m+1} converge to the true underlying value, then $u_{n-1}^{m+1,k+1}$ will be a better approximation of u_{n-1}^{m+1} than $u_{n-1}^{m+1,k}$. But then why not use this more precise value in the computation of $u_n^{m+1,k+1}$? This leads to the following scheme (we emphasized the changed upper index by making it bold):

$$\begin{cases} y_n^{m+1,k+1} = \frac{1}{1+\alpha}\left[Z_n^m + \frac{1}{2}\alpha\left(u_{n-1}^{m+1,\mathbf{k+1}} + u_{n+1}^{m+1,k}\right)\right] \\ u_n^{m+1,k+1} = u_n^{m+1,k} + (y_n^{m+1,k+1} - u_n^{m+1,k}) \end{cases}.$$

Using $u_{n-1}^{m+1,k+1}$ is advantageous not only because of increased convergence speed, but also because no additional memory is needed to store the new value; the new value can just overwrite the old one.

It turns out that it is possible to accelerate the convergence of the process by manipulating the size of the adjustment term. This yields the following scheme:

$$\begin{cases} y_n^{m+1,k+1} = \frac{1}{1+\alpha}\left[Z_n^m + \frac{1}{2}\alpha\left(u_{n-1}^{m+1,\mathbf{k+1}} + u_{n+1}^{m+1,k}\right)\right] \\ u_n^{m+1,k+1} = u_n^{m+1,k} + \omega(y_n^{m+1,k+1} - u_n^{m+1,k}) \end{cases},$$

where ω is a so-called relaxation parameter. Any value of ω between 0 and 2 will guarantee convergence in the limit; if the value chosen is less than 1, then we talk about "under-relaxation", if the value chosen is greater than 1, then we talk about "over-relaxation." For our scheme, over-relaxation speeds up the convergence of the solution.

In practice, we can not achieve convergence of values $u_n^{m+1,k+1}$ (in fact, nor would we want to; computing, say, the 20^{th} decimal from the value of an American option is not relevant, and can even be detrimental if a lot of computing resources are needed). Instead, we must set up an approximate criterion for convergence.

It is convenient - and simple - to examine the convergence of vector $U^{m+1,k}$ by the magnitude of the updates to it. The magnitude of the updates can be characterized, for example, by computing the 2-norm of the update vector:

$$\left\|U^{m+1,k+1} - U^{m+1,k}\right\|_2 = \sqrt{\sum_{n=1}^{N_x-1}(u_n^{m+1,k+1} - u_n^{m+1,k})^2}.$$

In practice, one often chooses a small value $\varepsilon > 0$, and considers convergence achieved when $\left\|U^{m+1,k+1} - U^{m+1,k}\right\|_2 < \varepsilon$.

Other norms, like the 1-norm or the ∞-norm, can be used. These norms lead to the following two inequalities, respectively:

$$\left\|U^{m+1,k+1} - U^{m+1,k}\right\|_1 = \sum_{n=1}^{N_x-1}\left|u_n^{m+1,k+1} - u_n^{m+1,k}\right| < \varepsilon,$$
$$\left\|U^{m+1,k+1} - U^{m+1,k}\right\|_\infty = \max_{n=\overline{1,N_x-1}}\left|u_n^{m+1,k+1} - u_n^{m+1,k}\right| < \varepsilon.$$

The method described above solves the equation $MU^{m+1} = b^m$ iteratively, starting from the guess $U^{m+1,0} = U^m$. This method is called the method of successive over-relaxation, and it could have been employed to determine the value of European calls and puts. However, this is not the problem we must solve.

0.3 Projected Successive Over-relaxation

We must solve the linear complementarity problem. We can do this easily by a very simple modification to the successive over-relaxation scheme.

Indeed, an examination of the linear complementarity formulation allows us to write the following scheme:

$$\begin{cases} y_n^{m+1,k+1} = \frac{1}{1+\alpha}\left[Z_n^m + \frac{1}{2}\alpha\left(u_{n-1}^{m+1,\mathbf{k+1}} + u_{n+1}^{m+1,k}\right)\right] \\ u_n^{m+1,k+1} = \max(u_n^{m+1,k} + \omega(y_n^{m+1,k+1} - u_n^{m+1,k}), g_n^{m+1}) \end{cases}.$$

If we make abstraction of the presence of the max function and its second argument, the scheme above is just the successive over-relaxation method which would iteratively solve the equation $MU^{m+1} = b^m$. This corresponds to an unexercised (i.e. European) put. When the price of the European put would "slip" under the value of the instantaneous payoff, then the max function intervenes, and pushes the the solution up.

One might think that a simpler solution is possible: solve equation $MU^{m+1} = b^m$ using any suitable method, but then modify the resulting solution so that if $u_n^{m+1} < g_n^{m+1}$, then the solution at the respective point would be replaced by g_n^{m+1}. This approach is actually incorrect, as all values x_i^{m+1} depend on each other through the system of equations; changing any of them in isolation will destroy the solution. This is why the max function operates on the function values during the iteration, and not after.

Do you know of any method where the new values x_i^{m+1} are obtained in isolation, i.e. they do not depend explicitly on each other? If yes, you can apply the simplification rejected above to this method.

Formally proving that the scheme of the projected successive over-relaxation converges to the right solution is non-trivial; we do not attempt it.

0.4 Putting Everything Together

Solving the linear complementarity problem in the transformed coordinates is now possible. These are the steps one must undertake to find a solution:

(a) Set up the linear complementarity problem in terms of the variable and function changes.

(b) Initialize the solution with the known values derived from the initial condition: $u_n^0 = g_n^0$, $0 \leqslant n \leqslant N_x$.

(c) For each later step $m+1$, $0 \leqslant m < N_t$, initialize the boundary values $u_0^{m+1} = g_0^{m+1}$ and $u_{N_x}^{m+1} = 0$; these will not change (put in other words, their successive "approximations" will always be the same). Starting with initial guesses $u_n^{m+1,0} = u_n^m$, use the method of projected successive over-relaxation until the norm of the vector of updates $U_n^{m,k+1} - U_n^{m,k}$ falls under a threshold of ϵ.

(d) Once the solution for the transformed problem is available, the variable and function changes can be undone, and the solution of the original problem can be found. All the issues identified earlier relating to the non-uniform grid induced in the original coordinate space (t, S) by the uniform grid in the transformed coordinate space are still applicable to this problem.

(e) If desired, one can find the early exercise frontier. Given a fixed m, the early exercise frontier at time τ_m will be found between points x_n and x_{n+1} in the transformed grid which have the property that $u_n^m = g_n^m$, and $u_{n+1}^m > g_{n+1}^m$. Of course, this does not fully determine the position of the frontier point, only that it is located within an interval of length δx. For more precision, a finer grid can be used. It is also possible to get a better position of the frontier point if interpolation is used to approximate the values of function u for values of x between x_n and x_{n+1}. The collection of the discrete frontier points can be used to reconstruct the entire early exercise frontier in the transformed space. By undoing the transformations, the early exercise frontier can be determined in the original coordinates (t, S).